B.Sc Part II

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Maths

Ques Show that the set of all positive rational numbers forms an abelian Group under the composition defined by $a*b = ab/2$

Sol^n Let Q, denote the set of all positive rational numbers.
We define an operation "*" on Q as follows

$$a*b = ab/2 \ \forall \ a,b \in Q.$$

To show Q is a group.

I   Closure property   Since for every $a, b \in Q+$, $(ab)/2$ is also in Q, therefore Q is closed.

e.g. $\frac{1}{2}, \frac{3}{4} \in (Q+)$

then $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8} \in (Q,+)$

II. Associativity. Let $a, b, c \in Q$ then

$$(a*b)*c = \frac{ab}{2} * c = \frac{abc}{4}$$

$$= \frac{a}{2} * \frac{bc}{2} = a*(b*c).$$

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Existence of identity – The number e will be the identity element if $e \in Q$ and if $e * a = a = a * e \;\forall a \in Q$

Now $e * a = a \Rightarrow \frac{ea}{2} = a$

$\Rightarrow e = 2$. as $a \in Q^+$
so $a \neq 0$

Now $2 \in Q$ and $2 * a = \frac{2a}{2} = a = a * 2$
$\forall a \in Q$

$\therefore$ 2 is the identity element.

Existence of inverse Let a be any element of Q_+,
If the number b is to be the inverse of a, then we must have
$b * a = e = 2 = ba/2 = 2 \Rightarrow b = 4/a$

Now $a \in Q \Rightarrow 4/a \in Q_+$.
we have $\frac{4}{a} * a = 4a/2a = 2$

$= a * 4/a$

Therefore $4/a$ is the inverse of a

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Thus each element of Q, is inversible.

Commutativity

Let a, b ∈ Q. Then
$$a * b = (ab/2) = ba/2 = b * a$$

Hence $(Q_+, *)$ is an abelian group.